Roll No. : ....

Total No. of Questions : **11**] [Total No. of Printed Pages : **4**

APF–2166

M.A./M.Sc. (Final) Examination, 2022

MATHEMATICS

Paper - VI

(Topology and Functional Analysis)

Time : 3 Hours] [*Maximum Marks : **100***

Section–A **(Marks : 2 × 10 = 20)**

Note :– Answer all *ten* questions (Answer limit **50** words). Each question carries **2** marks.

Section–B **(Marks : 4 × 5 = 20)**

Note :– Answer all *five* questions. Each question has internal choice (Answer limit **200** words). Each question carries **4** marks.

Section–C **(Marks : 20 × 3 = 60)**

Note :– Answer any *three* questions out of five (Answer limit **500** words). Each question carries **20** marks.

Section–A

1. (i) What do you mean by topological equivalence ? Define.

   (ii) Homeomorphism

   (iii) T_1-axiom

   (iv) Compact space

(v) Bounded linear transformation

(vi) Open mapping

(vii) Orthogonal complement

(viii) What does Hilbert space reflexive mean ? Define.

(ix) Unitary operator

(x) Adjoint operator

Section–B

2. Show that a subset of a topological space is open iff it is neighbourhood of each of its points.

Or

Let (Y, V) be a subspace of a topological space (X, T) and let (Z, W) be a subspace of (Y, V). Then show that (Z, W) is a subspace of (X, T).

3. Show that every topology finer than a T_1-topology on any set X is a T_1-topology.

Or

Show that every finite Hausdorff space is discrete.

4. Let N be a normed linear space and M be a subspace of N. Then show that the closure $\bar{M}$ of M is also a subspace of N.

Or

If N and N′ are normed linear spaces and T : N → N′. Then show that :

$$\|T\| = \sup\left\{\frac{\|Tx\|}{\|x\|},\ x \in N,\ x \neq 0\right\} \Leftrightarrow \|T\| = \sup\{\|Tx\|,\ x \in N \,\&\, \|x\| \leq 1\}$$

5. If X is an IPS, then show that $\|x\| = (x\ ,\ x)^{1/2}$ is a norm on X. (IPS–inner product space).

Or

If S is a non-empty subset of a Hilbert space H, then show that S^1 is a closed linear subspace of H.

6. Let T be an operator on a Hilbert space H. Then show that there exists a unique operator T* on H such that for all $x, y \in$ H, $(Tx, y) = (x, T^*y)$.

Or

If T is an operator on a Hilbert space H, then show that T is normal iff its real and imaginary parts commute.

Section–C

7. (i) Show that the mapping $f : (X, T) \to (X, T^*)$ is continuous iff $T^* \subset T$, (X, T), (X, T*) are topological spaces.

   (ii) In any topological space, show that :

$$\bar{A} = A \cup D(A)$$

8. (i) Show that a metric space is regular.

   (ii) Show that an open continuous image of a locally compact space is locally compact.

9. (i) Let N and N′ be normal linear spaces. Then show that N × N′ is a normed linear space with coordinate wise operations and the norm $\| (x, y) \| = \| x \| + \| y \|, x \in N, y \in N'$.

   (ii) Let M be a closed subspace of a normed linear space N and let x_0 be a vector not in M. If d is the distance of x_0 from M, then show that there exists a functional f_0 in N* such that :

$$f_0(M) = 0, f_0(x_0) = 1 \text{ and } \| f_0 \| = \frac{1}{d}$$

10. (i) If M is a closed linear subspace of a Hilbert space H, then show that :

$$H = M \oplus M^1$$

   (ii) If $\{e_i\}$ is an orthonormal set in a Hilbert space H and if x is any vector in H, then show that the set $S = \{e_i : (x, e_i) \neq 0\}$ is either empty or countable.

11. (i) Let H be a given Hilbert space and T* be the adjoint of operator T. Then show that T* is bounded linear transformation.

(ii) If $<T_n>$ is a sequence of self adjoint operators on a Hilbert space H and if $<T_n>$ converges to an operator T, then show that T is self adjoint.