Roll No. : ....

Total No. of Questions : **16**] [Total No. of Printed Pages : **4**

MATHSEM–118

M.A./M.Sc. (Ist Semester) Examination Dec., 2022

MATHEMATICS

Paper – II

(Advanced Complex Analysis)

*Time : **3** Hours*] [*Maximum Marks : **50***

The question paper contains three Sections.

Section–A **(Marks : 1 × 9 = 9)**

Note :– The candidate is required to answer all the *nine* questions carries **1** mark each. The answer should not exceed **50** words.

Section–B **(Marks : 4 × 5 = 20)**

Note :– The candidate is required to answer *five* questions by selecting at least *one* question from each Unit. Each question carries **4** marks. Answer should not exceed **200** words.

Section–C **(Marks : 7 × 3 = 21)**

Note :– The candidate is required to answer *three* questions by selecting *one* question from each Unit. Each question carries **7** marks. The answer should not exceed **500** words.

Section–A

1. (i) Write polar form of Cauchy-Riemann equations.

   (ii) State Cauchy's theorem for multiconnected domain.

   (iii) State Lioville's theorem.

   (iv) Define essential singular point.

   (v) Find the residue of $\dfrac{\cot \pi z}{(z-a)^2}$ at $z = 1$.

   (vi) Define residue at infinity.

   (vii) Define meromorphic function.

   (viii) Write argument principle.

   (ix) Define analytic continuation.

Section–B

Unit–I

2. Show that the function $f(z) = |xy|^{1/2}$ satisfies the Cauchy-Riemann equation at the origin but is not analytic at that point.

3. If $f(z)$ is continuous on a contour C of length l and $|f(z)| \le M$ for every point z on C, then prove that :

$$\left|\int_C f(z)dz\right| \le Ml$$

4. By considering the Laurent'z series for :

$$f(z) = \frac{1}{(1-z)(z-2)}$$

prove that :

$$\int_C f(z)dz = 2\pi i$$

where C is any closed contour within the annulus $1 < |z| < 2$.

Unit–II

5. Use method of contour integration to prove that :

$$\int_0^{2\pi} \frac{d\theta}{1+a^2-2a\cos\theta} = \frac{2\pi}{1-a^2}, \quad 0 < a < 1$$

6. Find out the zeroes and discuss the nature of singularities of :

$$f(z) = \frac{z-2}{z^2} \sin\frac{1}{z-1}$$

7. State and prove Cauchy's residue theorem.

Unit–III

8. If $a > e$, use Rouche's theorem to prove that the equation $e^z = az^n$ has n roots inside the circle $|z| = 1$.

9. Prove that there cannot be more than one different direct analytic continuations of an analytic function $f(z)$ in the same domain.

10. Show that the power series :

$$z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots\ldots\ldots$$

may be analytically continued to a wider region by means of the series :

$$\log 2 - \frac{1-z}{2} - \frac{1}{2}\left(\frac{1-z}{2}\right)^2 - \frac{1}{3}\left(\frac{1-z}{2}\right)^3 - \ldots\ldots\ldots\ldots$$

Section–C

Unit–I

11. If a function $f(z)$ is an analytic function within and on a closed contour C and a is any point lying in it, then prove that :

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz$$

12. State and prove maximum modulus principle.

Unit–II

13. If a is an isolated essential singular point of $f(z)$, then prove that for given any positive numbers δ, ε however small and any number b however large, there exists a point z in the circle $|z - a| < \delta$ for which $|f(z) - b| < \varepsilon$.

14. Evaluate :

$$\int_0^\infty \frac{dx}{x^4 + a^4}, \qquad a > 0$$

Unit–III

15. State and prove Rauche's theorem.

16. State and prove Schwarz reflection principle for analytic functions.