Roll No. : ....

Total No. of Questions : **11**] [Total No. of Printed Pages : **4**

APMA–365

M.A./M.Sc. (Previous) Examination, 2023

MATHEMATICS

Paper - II

(Analysis)

*Time : **3** Hours*] [*Maximum Marks : **100***

Section–A **(Marks : 2 × 10 = 20)**

Note :– Answer all *ten* questions (Answer limit **50** words). Each question carries **2** marks.

Section–B **(Marks : 4 × 5 = 20)**

Note :– Answer all *five* questions. Each question has internal choice (Answer limit **200** words). Each question carries **4** marks.

Section–C **(Marks : 20 × 3 = 60)**

Note :– Answer any *three* questions out of five (Answer limit **500** words). Each question carries **20** marks.

Section–A

1. (i) Define Lebesgue outer measure of an arbitrary set E.

   (ii) Define a measurable set.

   (iii) Prove that every square-summable function is summable.

   (iv) Define Lower Lebesgue-Darboux sum.

   (v) State Maximum Modulus principle.

(vi) State Schwarz lemma.

(vii) Define essential singularity.

(viii) State open mapping theorem.

(ix) Find the residue at infinity of the function $\frac{z}{(z-a)(z-b)}$.

(x) Define a complete analytic function. 2×10=20

Section–B

2. Prove that a set E is measurable if and only if E′ is measurable.

Or

Prove that a function f is measurable if and only if the set $\{x : f(x) < r\}$ is measurable for each rational number r.

3. Prove that every bounded measurable function f on a measurable set E is L–integrable on E.

Or

Let f be a non-negative measurable function defined on a measurable set E and C is a finite non-negative number, then prove that :

$$\int_E C\, f(x)\, dx = C\int_E f(x)\, dx$$

4. Let $f(z)$ be analytic within and on a circle C, given by $|z - a| = R$ and if $|f(z)| \leq M$ for every z on C, then prove that :

$$|f^{(n)}(a)| \leq \frac{Mn!}{R^n}$$

Or

By considering the Laurent's series for the function :

$$f(z) = \frac{1}{(1-z)(z-2)}$$

prove that :

$$\int_C f(z)\, dz = 2\pi i$$

where C is any closed contour within the annulus $1 < |z| < 2$.

5. Prove that the limit point of the poles of a function $f(z)$ is a non-isolated essential singularity.

Or

By using Rouche's theorem prove that the equation

$$e^z = a\, z^n \quad (a > e)$$

has n roots inside the circle $|z| = 1$.

6. State and prove Cauchy's residue theorem.

Or

Prove that there can not be more than one different direct analytic continuations of an analytic function $f(z)$ in the same domain.

Section–C

7. (a) Let $\{E_n\}$ be an increasing sequence of measurable sets, then prove that :

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n)$$

(b) Prove that the set of all real numbers in the open interval (0, 1) is uncountable. 10+10

8. (a) Let f be bounded and L-integrable on $[a, b]$. If $a < c < b$, then prove that f is L-integrable on $[a, c]$ and $[c, b]$ and also prove :

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

(b) Let $f(x) \in L_2$ and $g(x) \in L_2$, then prove that :

$$\int_a^b f(x)\,g(x)\,dx \le \left(\int_a^b f^2(x)\,dx\right)^{1/2} \left(\int_a^b g^2(x)\,dx\right)^{1/2}$$ 10+10

9. (a) If $f(z)$ is analytic within and on a closed contour C and a is any point within the C, then prove that :

$$f(a) = \frac{1}{2\pi i}\int_{C}\frac{f(z)}{z-a}\,dz$$

(b) State and prove the Liouville's theorem. 10+10

10. (a) If $z = a$ is an isolated singularity of $f(z)$ and $f(z)$ is bounded on some deleted neighbourhood of a, then prove that a is a removable singularity.

(b) State and prove the argument principle.

11. By method of contour integration prove that :

(i) $$\int_0^{2\pi}\frac{d\theta}{1+a^2-2a\cos\theta} = \frac{2\pi}{1-a^2}, (0 < a < 1)$$

(ii) $$\int_0^{\infty}\frac{dx}{x^4+a^4} = \frac{\pi}{a^3\sqrt{2}}, \quad (a > 0)$$ 10+10