Roll No. : ....

Total No. of Questions : **11**] [Total No. of Printed Pages : **4**

AFMA–266

M.A./M.Sc. (Final) Examination, 2023

MATHEMATICS

Paper – Opt.-I

(Generalized Hypergeometric Functions)

*Time : **3** Hours*] [*Maximum Marks : **100***

Section–A **(Marks : 2 × 10 = 20)**

Note :– Answer all *ten* questions (Answer limit **50** words). Each question carries **2** marks.

Section–B **(Marks : 4 × 5 = 20)**

Note :– Answer all *five* questions. Each question has internal choice (Answer limit **200** words). Each question carries **4** marks.

Section–C **(Marks : 20 × 3 = 60)**

Note :– Answer any *three* questions out of five (Answer limit **500** words). Each question carries **20** marks.

Section–A

1. (i) State Saalschütz theorem for the series ${}_3F_2$ with unit argument.

   (ii) Define generalized hypergeometric function.

   (iii) Write differential equation for ${}_pF_q$.

   (iv) State Bailey Theorem.

   (v) Define Meifer's G function with its condition of convergence.

(vi) Show that :

$$G_{01}^{10}\left[z\middle|_{0}^{-}\right]=e^{-z}$$

(vii) State any one property of H-function.

(viii) Define H-function.

(ix) What is contiguous function relation for H-function ? Explain briefly.

(x) Write any *one* recurrence relation for H-function.

Section–B

2. State and prove Dixon's theorem for the series ${}_3F_2$ with unit argument.

Or

State and prove Watson's theorem for the series ${}_3F_2$ with unit argument.

3. Show that :

$${}_1F_1(a,\, b,\, x)\; {}_1F_1(a,\, b, -x) = {}_2F_3\left(\begin{matrix} a, & b-a & \\ b, \dfrac{b}{2}, & \dfrac{b}{2}+\dfrac{1}{2} \end{matrix};\ \frac{x^2}{4}\right)$$

Or

Show that :

$$(a_p - a_1)G_{p,q}^{m,n}\left[z\middle|\begin{matrix} a_1,\, a_2,\, \ldots\ldots,\, a_p \\ b_1,\, b_2,\, \ldots\ldots,\, b_q \end{matrix}\right]$$

$$= G_{p,q}^{m,n}\left[z\middle|\begin{matrix} a_1-1,\, a_2,\, \ldots\ldots,\, a_p \\ b_1, \quad b_2,\, \ldots\ldots,\, b_q \end{matrix}\right]$$

$$+ G_{p,q}^{m,n}\left[z\middle|\begin{matrix} a_1,\, a_2,\, \ldots\ldots,\, a_{p-1},\, a_p-1 \\ b_1,\, b_2,\, \ldots\ldots,\, b_q \end{matrix}\right] \quad 1 \le n \le p-1$$

4. Prove that :

$$G_{p,q}^{m,n}\left[z\middle|\begin{matrix} a_1,\, a_2,\, \ldots\ldots,\, a_p \\ b_1,\, b_2,\, \ldots\ldots,\, b_{q-1}, a_1 \end{matrix}\right]$$

$$= G_{p-1,\, q-1}^{m,\, n-1}\left[z\middle|\begin{matrix} a_2,\, a_3,\, \ldots\ldots,\, a_p \\ b_1,\, b_2,\, \ldots\ldots,\, b_{q-1} \end{matrix}\right]$$

Or

Prove that :

$$z^k \frac{d^k}{dz^k}\left[G_{p,q}^{m,n}\left[z^{-1}\middle|\begin{matrix}(a_p)\\(b_q)\end{matrix}\right]\right] = (-1)^k G_{p+1,q+1}^{m,n+1}\left[z^{-1}\middle|\begin{matrix}1-k,\, a_p \\ b_q,\, 1\end{matrix}\right]$$

5. Show that :

$$z^{\sigma} H_{p,q}^{m,n}\left[z \left| \begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right.\right] = H_{p,q}^{m,n}\left[z \left| \begin{matrix}(a_p+\sigma A_p, A_p)\\(b_q+\sigma B_q, B_q)\end{matrix}\right.\right]$$

Or

Show That :

$$z^r \frac{d^r}{dz^r}\left[H_{p,q}^{m,n}\left[z^{-\delta} \left| \begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right.\right]\right.$$

$$= (-1)^r H_{p+1,q+1}^{m,n+1}\left[z^{-\delta} \left| \begin{matrix}(1-r,\delta),(a_p, A_p)\\(b_q, B_q)\ (1,\delta)\end{matrix}\right.\right]$$

6. Show that :

$$d(a_1 - 1, b_q)\ H(b_1+1) - d(b_q, b_1)\ H[a_1 - 1]$$
$$= d(b_1, a_1 - 1)\ H[bq + 1]$$

Or

Prove that :

$$\int_{-\pi/2}^{\pi/2} (\cos\theta)^{k+l-2} e^{i(k-l)\theta} H_{p,q}^{m,n}\left[z^{e^{ih\theta}} (\sec\theta)^h \left| \begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right.\right] d\theta$$

$$= \frac{\pi}{2^{k+l-2}\left|\overline{(k)}\right.} H_{p+1,q+1}^{m+1,n}\left[2^h z \left| \begin{matrix}(a_p, A_p),(l,h)\\(k+l-1,h),(b_q, B_q)\end{matrix}\right.\right]$$

Section–C

7. If neither $(a - b)$ nor $(a - c)$ nor a is a negative integer, then show that :

$${}_3F_2\left[\begin{matrix}a, b, c;\\1+a-b, 1+a-c;\end{matrix}\ x\right]$$

$$= (1-x)^{-a}\,{}_3F_2\left[\begin{matrix}\frac{a}{2}, \frac{a}{2}+\frac{1}{2}, 1+a-b-c;\\1+a-b, 1+a-c;\end{matrix}\ \frac{-4x}{(1-x)^2}\right]$$

8. If Re $(\alpha) > 0$ and Re$(\beta) > 0$ and if k and s are positive integers then inside the region of convergence of the resultant series :

$$\int_0^t x^{\alpha-1}(t-x)^{\beta-1}\,{}_pF_q\left(\begin{matrix}a_1, a_2, \ldots\ldots a_p\\b_1, b_2, \ldots\ldots b_q\end{matrix}; cx^k (t-x)^s\right) dx$$

$$= B(\alpha, \beta) t^{\alpha+\beta-1} {}_{p+k+s}F_{q+k+s}$$

$$\left[\begin{matrix} a_1, a_2, \ldots\ldots a_p, \frac{\alpha}{k}, \frac{\alpha+1}{k}, \ldots\ldots \frac{\alpha+k-1}{k}, \frac{\beta}{s}, \frac{\beta+1}{s}, \ldots\ldots \frac{\beta+s-1}{s}; \\ b_1, b_2, \ldots\ldots b_q \frac{\alpha+\beta}{k+s}, \frac{\alpha+\beta+1}{k+s}, \ldots\ldots, \frac{\alpha+\beta+k+s-1}{k+s}; \end{matrix} \frac{k^k \cdot s^s (t)^{k+s}}{(k+s)^{k+s}}\right]$$

9. (i) Prove that :

$$2\pi i \, G^{m,n}_{p,q}\left[z \left| \begin{matrix}(a_p)\\(b_q)\end{matrix}\right.\right]$$

$$= \exp(i\pi b_{m+1}) G^{m+1,n}_{p,q}\left[ze^{-i\pi} \left| \begin{matrix}(a_p)\\(b_q)\end{matrix}\right.\right]$$

$$- \exp(i\pi b_{m+1}) G^{m+1,n}_{p,q}\left[ze^{i\pi} \left| \begin{matrix}(a_p)\\(b_q)\end{matrix}\right.\right]$$

(ii) Evaluate :

$$G^{1,2}_{2,2}\left[z \left| \begin{matrix}\frac{5}{2}; -1\\ 0, \frac{5}{2}\end{matrix}\right.\right]$$

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10. Show that :

$$\left(\frac{d}{dx} x - c_1\right) \ldots\ldots \left(\frac{d}{dx} \cdot x - c_r\right) \left\{ x^s H^{m,n}_{p,q}\left[zx^h \left| \begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right.\right]\right.$$

$$= x^s H^{m,n+r}_{p+r,q+r}\left\{zx^h \left| \begin{matrix}(c_r - s - 1, h), (a_p, A_p)\\(b_q, B_q), \quad (c_s - s, h)\end{matrix}\right.\right]$$

11. Show that :

$$H^{m,n}_{p,q}\left[\eta w \left| \begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right.\right] = \eta^{(a_1-1)/A_1} \sum_{r=0}^{\infty} \frac{\left(1-\eta^{-1/A_1}\right)^r}{\lfloor r}$$

$$\times H^{m,n}_{p,q}\left[w \left| \begin{matrix}(-r + a_1, A_1, (a_p, A_p)\\(b_q, B_q)\end{matrix}\right.\right]$$