Roll No. : ....

Total No. of Questions : **11**] [Total No. of Printed Pages : **4**

AFMA–264

M.A./M.Sc. (Final) Examination, 2023

MATHEMATICS

Paper – VI

(Topology and Functional Analysis)

Time : 3 Hours] [*Maximum Marks : 100*

Section–A **(Marks : 2 × 10 = 20)**

Note :– Answer all *ten* questions (Answer limit **50** words). Each question carries **2** marks.

Section–B **(Marks : 4 × 5 = 20)**

Note :– Answer all *five* questions. Each question has internal choice (Answer limit **200** words). Each question carries **4** marks.

Section–C **(Marks : 20 × 3 = 60)**

Note :– Answer any *three* questions out of five (Answer limit **500** words). Each question carries **20** marks.

Section–A

1. (i) Define coarser and finer topologies.

   (ii) Define continuity in topoloigcal space.

   (iii) Define Hausdorff space.

   (iv) Define sequentially compact space.

(v) Define Banach space.

(vi) State closed graph theorem.

(vii) State parallelogram law.

(viii) Define orthonormal set.

(ix) Define self adjoint operator.

(x) Define projection.

Section–B

2. Let $\{T_\lambda : \lambda \in \wedge\}$, where $\wedge$ is an arbitrary set, be a collection of topologies for X. Then show that the intersection $\cap\{T_\lambda : \lambda \in \wedge\}$ is also a topology for X.

Or

Let (X, d) be a metric space. Show that the mapping $d^* : X \times X \to R$ defined by :

$$d^*(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is also a metric on X and that d and d^* are equivalent.

3. If A is an infinite subset of a compact space X, then show that A has limit point in X.

Or

Show that a topological space (X, J) is a T_1-space if and only if every singleton subset $\{x\}$ of X is J-closed.

4. In a normed linear space, show that every convergent sequence is a Cauchy sequence.

Or

Let M be a closed linear subspace of a normed linear space N. For each coset x + M in the quotient space N/M, define $\| x + M \| = \inf \{\| x + m \| : m \in M\}$. Then show that N/M is a normed linear space.

5. If x, y are any two vectors in a Hilbert space H, then show that $|(x, y)| \leq \| x \| \| y \|$.

Or

Show that a closed convex subset of C of a Hilbert space H contains a unique vector of smallest norm.

6. If T is an operator on a Hilbert space H, then show that $(Tx, x) = 0$ for all $x \in H$ if and only if $T = 0$ (zero operator).

Or

If N_1 and N_2 are normal operators on a Hilbert space H with the property that either commutes with the adjoint of the other, then show that $N_1 + N_2$ and N_1N_2 are also normal operators.

Section–C

7. (i) Let $F_1, F_2, \ldots\ldots F_n$ be a finite number of closed subsets of a topological space X. Then show that their union will also be a closed subset of X.

   (ii) Show that Homeomorphism is an equivalence relation in the collection of all topological spaces.

8. (i) Show that a metric space is compact if and only if it is complete and totally bounded.

   (ii) Let X be a topological space and $x \in X$. Then show that X is regular if and only if the collection of all closed neighbourhoods of x forms a local base at x.

9. (i) State and prove open mapping theorem.

   (ii) Let T be a linear transformation of a normed linear space N into another normed linear space N. Then show that the following statements are equivalent :

(a) T is continuous

(b) T is continuous at the origin

(c) There exists a real number $k \geq 0$ such that :

$$\| T(x) \| \leq k \| x \| \text{ for all } x \in N$$

10. (i) Let $\{e_1, e_2, ...e_n\}$ be a finite orthonormal set in a Hilbert space H. If x is any vector in H, then show that :

$$\sum_{i=1}^{n} |(x, e_i)|^2 \leq \| x \|^2$$

and $x - \sum_{i=1}^{n} |(x, e_i)\, e_i \perp e\, j.$ for each j.

(ii) Let H be a Hilbert space, and let f be an arbitrary functional in H^*. Then show that there exists a unique vector y in H such that :

$$f = f_y \ ie\ f(x) = (x, y)\ \forall\ x \in H.$$

11. (i) Show that an operator T on a Hilbert space H is unitary if and only if it is an isometric isomorphism of H onto itself.

(ii) If P is a projection on a Hilbert space H with range M and null space N, then show that $M \perp N$ if and only if P is self adjoint, and in this case $N = M^{\perp}$.