Roll No : ....

Total No. of Questions : **11**] [Total No. of Printed Pages : **4**

ASP–640

M.A./M.Sc. (Final) Examination, 2021

MATHEMATICS

Paper - VI

(Topology and Functional Analysis)

Time : 1½ Hours] [*Maximum Marks : **100***

Section–A **(Marks : 2 × 10 = 20)**

Note :– Answer all *ten* questions. (Answer limit **50** words). Each question carries **2** marks.

(खण्ड–अ) **(अंक : 2 × 10 = 20)**

नोट :– सभी **दस** प्रश्नों के उत्तर दीजिए। (उत्तर-सीमा **50** शब्द)। प्रत्येक प्रश्न **2** अंक का है।

Section–B **(Marks : 4 × 5 = 20)**

Note :– Answer all *five* questions. Each question has internal choice (Answer limit **200** words). Each question carries **4** marks.

(खण्ड–ब) **(अंक : 4 × 5 = 20)**

नोट :– सभी **पाँच** प्रश्नों के उत्तर दीजिए। प्रत्येक प्रश्न में विकल्प का चयन कीजिए (उत्तर-सीमा **200** शब्द)। प्रत्येक प्रश्न **4** अंक का है।

Section–C **(Marks : 20 × 3 = 60)**

Note :– Answer any *three* questions out of five (Answer limit **500** words). Each question carries **20** marks.

(खण्ड–स) **(अंक : 20 × 3 = 60)**

नोट :– पाँच में से किन्हीं **तीन** प्रश्नों के उत्तर दीजिए (उत्तर-सीमा **500** शब्द)। प्रत्येक प्रश्न **20** अंक का है।

Section–A 2 each

1. Define the following :

   (i) Coarser topology

   (ii) Relative topology

   (iii) Hausdorff space

   (iv) Regular space

   (v) Open mapping

   (vi) State closed graph theorem

   In a complex inner product space X, show that :

   (vii) $(x, \beta y + \gamma z) = \overline{\beta}\ (x, y) + \overline{\gamma}\ (x, z),\ x, y, z \in X$.

   (viii) $(x, 0) = 0,\ \forall\ x \in X$

   Define with respect to Hillbert space :

   (ix) Normal operator

   (x) Projection

Section–B 4 each

2. If C is a base for the topological space X and Y is a subspace of X, then show that $A = \{B \cap Y : B \in C\}$ is a base for Y.

   Or

   Let X and Y be topological spaces and $f : X \to Y$. Then show that f is continuous iff $f^{-1}(B^0) = [f^{-1}(B)]^0$ for every subset B of Y.

3. Show that the property of a space being a T_0-space is hereditary.

   Or

   Show that the property of being a T_1-space is a topological property.

4. Show that the subset M of a normed linear space N is bounded if and only if there is a positive constant K such that :

$$\|x\| \le K\ \forall\ x \in M$$

Or

Let T : N → N′ be a linear transformation. Then show that T is a bounded if and only if T maps bounded sets in N into bounded sets in N′. where N and N′ are normed linear spaces over the same field.

5. If S is a non-empty subset of a Hilbert space H, then show that $S^{\perp}$ is a closed linear subspace of H. Is $S^{\perp}$ a Hilbert space ? Why ? 3+1=4

Or

Let $\{e_1, e_2, \ldots\ldots\ldots, e_n\}$ be a finite orthonormal set in a Hilbert space H. If x is any vector in H, then show that :

$$\sum_{i=1}^{n} \left| (x, e_1') \right|^2 \Sigma \| x \|^2$$

6. In a Hilbert space H, show that :

$$\| T^* T \| = \| T \|^2$$

where T* is adjoint of operator T.

Or

If $\{T_n\}$ is a sequence of self adjoint operators on a Hilbert space H and if $\{T_n\}$ converges to an operator T, then show that T is self adjoint.

Section–C 20 each

7. Let X and Y be topological spaces and f : X → Y be a bijective map. Show that the following are equivalent :

(i) f is open and continuous

(ii) f is homeomorphism

(iii) f is closed and continuous

8. Show that in a metric space (X, d) the following are equivalent :

   (i) X is compact

   (ii) X is limit point compact

   (iii) X is sequentially compact

9. Let M be a linear subspace of normed linear space N and f be a functional defined on M. If x_0 is a vector not in M and if M_0 is the subspace spanned by M and x_0, then show that f can be extended to a functional f_0 defined on M_0 such that :

$$\|f_0\| = \|f\|$$

10. Let H be a Hilbert space and f be an arbitray functional in H^*. Then show that there exists a unique vector y in H such that :

$$f(x) = (x, y) \ \forall \ x \in \text{H and } \|f\| = \|y\|$$

11. Let H be a given Hilbert space and T* be adjoint of the operator T on H. Then show that T* is a bounded linear transformation and T determines T* uniquely.