Roll No. : ....

Total No. of Questions : **11**] [Total No. of Printed Pages : **4**

APG–1075

M.A./M.Sc. (Previous) Examination, 2021

MATHEMATICS

Paper - II

(Analysis)

Time : 1½ Hours] [*Maximum Marks : **100***

Section–A **(Marks : 2 × 10 = 20)**

Note :– Answer all *ten* questions (Answer limit **50** words). Each question carries **2** marks.

Section–B **(Marks : 4 × 5 = 20)**

Note :– Answer all *five* questions. Each question has internal choice (Answer limit **200** words). Each question carries **4** marks.

Section–C **(Marks : 20 × 3 = 60)**

Note :– Answer any *three* questions out of five (Answer limit **500** words). Each question carries **20** marks.

Section–A 2 each

1. (i) Show that two countable sets are equivalent.

   (ii) State Weierstrass' theorem on the approximation of continuous function by polynomials.

(iii) Define Lebesque integral function.

(iv) Write the conditions of the space L_2 of square summable functions is a linear space.

(v) Write any *two* properties of complex integrals.

(vi) Write down the statement of Laurent's theorem.

(vii) Find the poles of the function :

$$f(z) = \frac{z+2}{(z+1)^2(z-2)}$$

(viii) Define removable singularity.

(ix) Evaluate residue at $z = 1$ for the function :

$$f(z) = \frac{z^2}{(z-1)(z-2)(z-3)}$$

(x) State Rouche's theorem.

Section–B 4 each

2. Prove that if E_1 and E_2 are measureable sets, then $E_1 \cup E_2$ is also measurable.

Or

Prove that a continuous function defined on a measurable set E is measurable but the converse need not be true.

3. Prove that every bounded measurable function f on a measurable set E is L-integrable on E.

Or

Let f be a summable on a set E, then show that for given each $\in > 0$, there exists a $\delta > 0$ such that :

$$\left| \int_e f(x)\, dx \right| < e$$

where e is a measurable subset of E with $m(e) < \delta$.

4. Evaluate :

$$\int_{|z|=1} \frac{\sin^6 z}{(z-\pi/6)^3} dz$$

Or

Expand $\frac{1}{z(z^2-3z+2)}$ in Laurent's series for the regions :

(i) $0 < |z| < 1$

(ii) $|z| > 2$

5. Show that the function $f(z) = e^{-1/z^2}$ has no singularity. 2+2=4

Or

Determine the nature of the pole at origin of the function :

$$\frac{e^z}{z \sin mz}$$

6. Find the residue of $\frac{z^3}{z^2-1}$ at $z = \infty$.

Or

Show that the function :

$$f(z) = 1 + z + z^2 + ... = \sum_{n=0}^{\infty} z^n$$

can be continued analytically outside the circle of convergence.

Section–C

7. (i) Let $\{E_n\}$ be a countable collection of sets, then prove that :

$$m^*\left(\bigcup_n E_n\right) \leq \sum_n m^*(E_n)$$

(ii) Let f be a continuous function defined on the closed interval $[a, b]$, then show that for each $\in > 0$ there exists a polynomial $p(x)$ such that :

$$|f(x) - p(x)| < \in, \text{ for all } x \in [a, b]$$

10+10=20

8. Let f be a bounded measurable function defined on a measurable set E and E be the union of a countable class $\{E_i : i \in N]$ of pairwise disjoint sets, then prove that :

$$\int_{E} f(x)\, dx = \sum_{i=i}^{\infty} \int_{E_i} f(x)\, dx$$ 20

9. (i) State and prove Taylor's theorem.

(ii) Expand $f(z) = e^z$ by Taylor's theorem about the point $z = 0$. 16+4=20

10. (i) Define Branch point.

(ii) State and prove Riemann's theorem on removable singularity. 4+16=20

11. By the method of contour integration, show that :

$$\int_0^{\infty} \frac{\cos mx}{x^2 + a^2} dx = \frac{\pi e^{-ma}}{2a}$$

$m \geq 0$, $a > 0$. Hence deduce that :

(i) $\int_0^{\infty} \frac{\cos mx}{1 + x^2} dx = \frac{\pi}{2} e^{-m}, m \geq 0$

(ii) $\int_{-\infty}^{\infty} \frac{\cos x}{a^2 + x^2} dx = \frac{\pi}{2} e^{-a}$ 16+2+2=20