Roll No. : ....

Total No. of Questions : **11**] [Total No. of Printed Pages : **4**

APG–1077

M.A./M.Sc. (Previous) Examination, 2021

MATHEMATICS

Paper - IV

(Differential and Integral Equations)

Time : 1½ Hours] [*Maximum Marks : **100***

Section–A **(Marks : 2 × 10 = 20)**

Note :– Answer all *ten* questions (Answer limit **50** words). Each question carries **2** marks.

Section–B **(Marks : 4 × 5 = 20)**

Note :– Answer all *five* questions. Each question has internal choice (Answer limit **200** words). Each question carries **4** marks.

Section–C **(Marks : 20 × 3 = 60)**

Note :– Answer any *three* questions out of five. (Answer limit **500** words). Each question carries **20** marks.

Section–A 2 each

1. (i) If the region S be the rectangle $|x| \leq a$ and $|y| \leq b$, show that the function :

$$f(x, y) = x \sin y + y \cos x$$

satisfies the Lipschitz condition. Find the Lipschitz constant.

(ii) Classify the following P.D.E. :

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$$

(iii) Solve the BVP :

$$\frac{d^2 y}{dx^2} + y = 0$$

$y(0) = 0$, $y(1) = 0$.

(iv) Define Rayleigh quotient.

(v) Write one alternative form of Euler–Lagrange equation.

(vi) Test for an extremum the functional :

$$I = [y(x)] = \int_0^1 (xy + y^2 - 2y^2 y')\, dx \; , \; y(0) = 1, \; y(1) = 2$$

(vii) Define Singular Integral equation.

(viii) Find the iterated kernels $k_1(x, t)$ and $k_2\ (x, t)$, if :

$$K(x, t) = \sin (x - 2t); \; a = 0, \; b = 2\pi$$

(ix) Define Volterra Integral equation.

(x) Convert the following Volterra integral equation of first kind into the second kind :

$$\int_0^x e^{x-t} g(t)\, dt = \sin x$$

Section–B 4 each

2. Find the largest interval $|x| \le a$ at which the following IVP has a unique solution :

$$\frac{dy}{dx} = e^y, \; y(0) = 0$$

Or

Solve BVP :

$$\frac{\partial u}{\partial x} = 4 \frac{\partial u}{\partial y}$$

if $u(0, y) = 8e^{-3y}$.

3. Find the eigenvalues and eigenfunction of the BVP :

$$y'' + \lambda y = 0,\ y(0) + y'(0) = 0,\ y'(1) = 0$$

Or

Use Green's function technique to obtain the solution of :

$$\frac{d^2 y}{dx^2} = f(x),$$

in $0 \le x \le 1$, subject to the BCs : $y(0) = y\ (1) = 0$.

4. Find the extremals of the functional :

$$\int_a^b \frac{y'^2}{x^3}\, dx$$

Or

Find the curve of fixed length L that joins the points (0, 0) and (1, 0), lies above the x-axis, and encloses the maximum area between itself and the x-axis.

5. For what value of λ the function $g(x) = 1 + \lambda x$ is a solution of the integral equation :

$$x = \int_0^x e^{x-t}\, g(t)\, dt$$

Or

Solve :

$$g(x) = 1 + \int_0^1 (1 + e^{x+t})\, g(t)\, dt$$

6. Find the resolvent Kernel for Volterra integral equation with the following kernel :

$$k(x,\ t) = e^{x-t}$$

Or

Prove that the eigenvalues of a symmetric kernel are real.

Section–C

7. (a) Reduce the equation :

$$y^2 \frac{\partial^2 z}{\partial x^2} - 2xy \frac{\partial^2 z}{\partial x\, \partial y} + x^2 \frac{\partial^2 z}{\partial y^2} = \frac{y^2}{x} \frac{\partial z}{\partial x} + \frac{x^2}{y} \frac{\partial z}{\partial y}$$

to canonical form and hence solve it.

(b) Solve one dimensional Heat equation :

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{k} \frac{\partial z}{\partial t}$$

where $z = z(x, t)$ with BCs : $z(0, t) = z(1, t) = 0$ for all t. 10,10

8. (a) Find an expansion for $f(x) = e^x$ in terms of the eigenfunctions of the Sturm–Liouville BVP :

$$y'' + \lambda y = 0,\ y'(0) = 0,\ y(n) = 0$$

(b) Find modified Green's function for the equation $y'' = f(x)$ with BCs :

$$y'(0) = 0,\ y'(L) = 0$$ 10,10

9. (a) Find the path on which a particle in the absence of friction, will slide from one fixed point to another point in the shortest time under the action of gravity.

(b) Find the extremum curves for the functional :

$$I[y(x)] = \int_0^{x_2} \frac{\sqrt{1+y'^2}}{y}\, dx$$

given that $y(0) = 0$ and $y_2 = x_2 + 5$. 10,10

10. (a) Convert the following BVP into an integral equation :

$$\frac{d^2 y}{dx^2} + xy = 1;\ y(0) = 0 = y(1)$$

(b) Solve the following integral equation by the method of resolvent kernel :

$$g(x) = \left(\sin x - \frac{\pi}{4}\right) + \frac{1}{4}\int_0^{\pi/2} xt\, g(t)\, dt$$ 10,10

11. (a) Solve the following integral equation by Fredholm theory :

$$g(x) = 1 + \int_0^1 (1 - 3xt)\ g(t)\, dt$$

(b) Solve the following Volterra integral equations, with the aid of resolvent kernel :

$$g(x) = 1 + x^2 + \int_0^x \frac{1+x^2}{1+t^2}\, g(t)\, dt$$ 10,10